If we have $(D - \alpha)^2 y = 0$, we may therefore get $(D - \alpha)y = c_1 e^{\alpha x}$. On repeating this operation,

$$y = e^{\alpha x} \int e^{-\alpha x} c_1 e^{\alpha x} dx = c_1 x e^{\alpha x} + c_0 e^{\alpha x}.$$

If we have $(D - \alpha)^3 y = 0$, we get in precisely the same way

$$y = c_2 x^2 e^{\alpha x} + c_1 x e^{\alpha x} + c_0 e^{\alpha x}.$$

The complementary function would then be completely known on proving the usual theorem that if $v_1, v_2, \cdots v_n$ are solutions of $f(D)y = 0$ then $c_1 v_1 + c_2 v_2 + \cdots c_n v_n$ is also a solution.

One need hardly draw attention to the fact that the complete value of y will be obtained if the factors of

$$(D - \alpha_1)(D - \alpha_2) \cdots (D - \alpha_n)y = X$$

are removed in turn by the above rule, the constant of integration being introduced each time.

A GRAPHICAL SOLUTION OF THE DIFFERENTIAL EQUATION OF THE FIRST ORDER.*

By T. R. RUNNING, University of Michigan.

The following graphical solution of the differential equation of the first order was derived while trying to solve a problem involving two such equations. As an illustration, these equations will be solved after a discussion of the method of solution has been given.

If the primitive of a differential equation which expresses the condition of a problem can not be found, or can be found with difficulty, it may i e desirable to have a method of approximate solution which does not involve a great deal of labor. It is thought that the method given is more easily carried out than any other which has come to the writer's attention.[1]

The solution depends upon the simple fact that the area under the derived curve is equal to the corresponding ordinate of the integral curve. In Fig. 1

$$\text{Area } OPQ = RQ.$$

If in the equation

$$\frac{dy}{dx} = f(x, y)$$

we assign particular values to y and plot the resulting equations with dy/dx as

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* First part of a paper presented to the American Mathematical Society at Madison, September 9, 1913.

[1] For a comparison of approximate methods see "Beitrag zur näherungsweisen Integration totaler Differentialgleichungen," by W. Kutta, *Zeitschrift für Mathematik und Physik*, Vol. 46, pp. 435–453.

ordinate and x as abscissa, we shall have, designating each curve by the corresponding value of y, a family of curves as shown in Fig. 2.

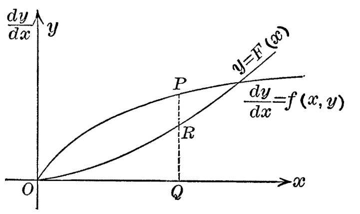

FIG. 1.

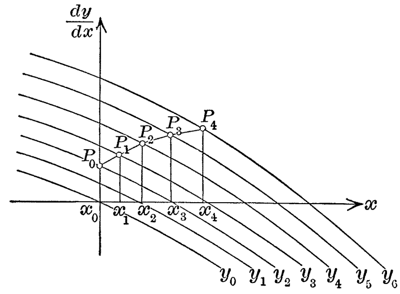

FIG. 2.

Suppose the initial condition to be $y = y_2$ when $x = 0$. Draw a curve, which for an approximation may be a straight line, from P_0, where the curve y_2 crosses the axis of dy/dx, to the curve y_3, such that the area under it shall be equal to $y_3 - y_2$. From this point, P_1, draw a line to a point P_2 on the curve y_4 such that the area under the line P_1P_2 shall be equal to $y_4 - y_3$. This process continued will give any desired number of points on the integral curve. The curve connecting the points $P_0P_1 \cdots P_n$ is the derived curve.

It is to be noted that the integral curve is not obtained by graphically integrating the derived curve, but the coördinates of its points are read from the figure. Thus (x_0, y_2), (x_1, y_3), (x_2, y_4), (x_3, y_5), etc., are points which determine the integral curve. If a close approximation is required the values of y must be so chosen that the corresponding curves will lie near together.

The problem mentioned was one which gave rise to the two differential equations

$$(1) \qquad \frac{dy}{dx} = \frac{1800x - 132y^{\frac{3}{2}}}{11y^{\frac{6}{5}} + 1000},$$

$$(2) \qquad \frac{dy}{dx} = \frac{1740 - 240x - 132y^{\frac{3}{2}}}{11y^{\frac{6}{5}} + 1000}.$$

It was required to find the maximum value of y and the corresponding value of x, knowing that when $x = 0$, $y = 0$, equation (1) was to be satisfied between $x = 0$ and $x = \frac{29}{34}$, equation (2) between $x = \frac{29}{34}$ and $x = 7\frac{1}{4}$. There was another equation to be satisfied beyond $x = 7\frac{1}{4}$ but, as the conditions clearly showed, the maximum value of y occurs before x reaches the value $7\frac{1}{4}$, it need not be considered.

The curves which correspond to different values of y are straight lines and shown in Fig. 3.

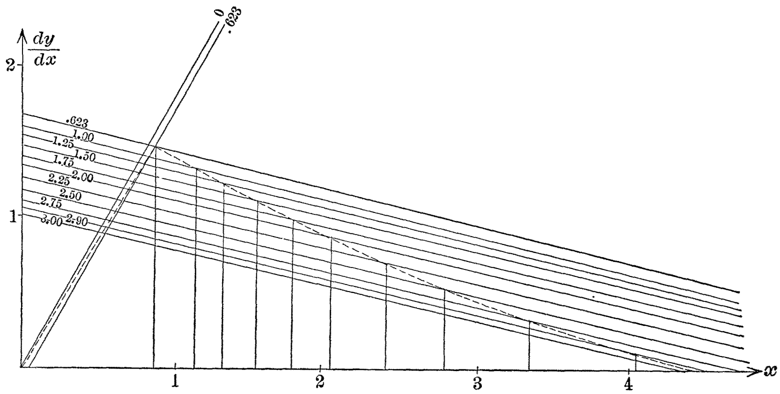

FIG. 3.

y will be a maximum when $dy/dx = 0$. This value is 2.926 and the corresponding value of x is 4.492. The values chosen for y are indicated on the figure. It is seen that when $x = \frac{29}{34}$, $y = .623$. This value must, therefore, be used in both differential equations.

The method was applied to the problem given in Kutta's article, above referred to, for comparing the five different approximations. With squared paper ruled to twentieths of an inch and using two inches as the unit, the approximation was closer than that of any one of the first four but not quite so close as Kutta's, while the work necessary to obtain the result was much less.

In a later paper it is proposed to extend the method to the solution of the differential equation of the second order.

BOOK REVIEWS.

W. H. BUSSEY, CHAIRMAN OF THE COMMITTEE.

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